ON A CLASS OF II_1 FACTORS WITH AT MOST ONE CARTAN SUBALGEBRA II

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ABSTRACT. This is a continuation of our previous paper studying the structure of Cartan subalgebras of von Neumann factors of type II_1. We provide more examples of II_1 factors having either zero, one or several Cartan subalgebras. We also prove a rigidity result for some group measure space II_1 factors.

1. INTRODUCTION

A celebrated theorem of Connes ([Co]) states that all amenable II_1 factors are isomorphic to the approximately finite dimensional II_1 factor R of Murray and von Neumann. In particular, all group II_1 factors $L(\Gamma)$ associated with ICC (infinite conjugacy class) amenable groups Γ, and all group measure space II_1 factors $L^\infty(X) \rtimes \Gamma$ arising from (essentially) free ergodic probability-measure-preserving (abbreviated as p.m.p.) actions $\Gamma \curvearrowright X$ of countable amenable groups Γ on standard probability spaces X, are isomorphic to R.

In contrast to the amenable case, the group measure space II_1 factors $L^\infty(X) \rtimes \Gamma$ of free ergodic p.m.p. actions of non-amenable groups Γ on standard probability spaces X form a rich and particularly important class of II_1 factors. More general crossed product construction provides a wider class. We want to investigate the isomorphy problem of the crossed product II_1 factors. Namely, given the crossed product $M = Q \rtimes \Gamma$ of a finite amenable von Neumann algebra (Q, τ) by a τ-preserving action of a countable group Γ, to what extent can we recover information on the original action $\Gamma \curvearrowright Q$? In particular, does there exist a group measure space II_1 factor $M = L^\infty(X) \rtimes \Gamma$ which remembers completely the group Γ and the action $\Gamma \curvearrowright X$? The first task would be to determine all regular amenable subalgebras of a given II_1 factor M. Recall that a von Neumann subalgebra P of M is said to be *regular* if the normalizer group of P in M generates M as a von Neumann algebra ([Di]). A regular maximal abelian subalgebra A of M is called a *Cartan subalgebra* ([FM]). In the case of a group

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measure space II_1 factor $M = L^\infty(X) \rtimes \Gamma$, the von Neumann subalgebra $L^\infty(X)$ is a Cartan subalgebra and determining its position amounts to recovering the orbit equivalence relation of the original action $\Gamma \curvearrowright X$ (see [FM]). By [CFW], the approximately finite dimensional II_1 factor R has a unique Cartan subalgebra, up to conjugacy by an automorphism of R. In the previous paper ([OP]), we provided the first class of examples of non-amenable II_1 factors having unique Cartan subalgebra. They are the group measure space II_1 factors $M = L^\infty(X) \rtimes \mathbb{F}_r$ associated with free ergodic p.m.p. profinite actions $\mathbb{F}_r \curvearrowright X$ of free groups $\mathbb{F}_r$. In this paper, we extend this result from the free groups $\mathbb{F}_r$ to a lager class of countable groups with the property (strong) $(\mathrm{HH})^+$, defined as follows.

Definition. Let G be a second countable locally compact group. By a 1-*cocycle*, we mean a continuous map $b\colon G \to \mathcal{K}$, or a triplet $(b, \pi, \mathcal{K})$ of b and a continuous unitary G-representation π on a Hilbert space $\mathcal{K}$, which satisfies the 1-cocycle identity:

$$\forall g, h \in \Gamma, \quad b(gh) = b(g) + \pi_g b(h).$$

The 1-cocycle b is called *proper* if the set $\{g \in G : \|b(g)\| \le R\}$ is compact for every $R > 0$. Assume that G is non-amenable. We say *G has the Haagerup property* (see [CJV, BO]) if it admits a proper 1-cocycle $(b, \pi, \mathcal{K})$. In the case when π can be taken non-amenable (resp. to be weakly contained in the regular representation), we say *G has the property (resp. strong)* (HH). We say *G has the property (strong)* $(\mathrm{HH})^+$ if Γ has the property (strong) (HH) and the complete metric approximation property (i.e., it is weakly amenable with constant 1).

In Section 2, we will prove that lattices of products of $\mathrm{SO}(n, 1)$ $(n \ge 2)$ and $\mathrm{SU}(n, 1)$ have the property $(\mathrm{HH})^+$, and that lattices of $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{C})$ have the property strong $(\mathrm{HH})^+$. Building on our previous work ([OP]) and Peterson's deformation technology ([Pe]), we obtain the following.

Theorem A. *Let $M = Q \rtimes \Gamma$ be the crossed product of a finite von Neumann algebra (Q, τ) by a τ-preserving action of a countable group Γ with the property* (HH). *Let $P \subset M$ be a regular weakly compact von Neumann subalgebra. Then, $P \preceq_M Q$.*

Since $L^\infty(X) \rtimes \Gamma$ has the complete metric approximation property if Γ has it and the action is profinite, the weak compactness assumption holds automatically (see Section 3).

Corollary A. *Let Γ be a countable group with the property* $(\mathrm{HH})^+$. *Then, $L(\Gamma)$ has no Cartan subalgebra. Moreover, if $\Gamma \curvearrowright X$ is a free ergodic p.m.p. profinite action, then $L^\infty(X)$ is the unique Cartan subalgebra in $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy.*

As in [OP], a stronger result holds if Γ has the property strong (HH).

Theorem B. *Let $M = Q \rtimes \Gamma$ be the crossed product of a finite amenable von Neumann algebra (Q, τ) by a τ-preserving action of a countable group Γ with the property strong* (HH). *Let $P \subset M$ be an amenable von Neumann subalgebra such that $P \not\preceq_M Q$ and $\mathcal{G} \subset \mathcal{N}_M(P)$ be a subgroup whose action on P is weakly compact. Then, the von Neumann subalgebra $\mathcal{G}''$ is amenable.*

Corollary B. *Let Γ be a countable group with the property strong* $(\mathrm{HH})^+$. *Then, $L(\Gamma)$ is strongly solid, i.e., the normalizer of every amenable diffuse subalgebra generates an amenable von Neumann subalgebra.*

Once the Cartan subalgebra $L^\infty(X)$ is determined, the isomorphy problem of $M = L^\infty(X) \rtimes \Gamma$ reduces to that of the orbit equivalence relations. Then, the group Γ and the action $\Gamma \curvearrowright X$ can be recovered if the orbit equivalence cocycle untwists ([Zi]). Ioana ([Io]) proved a cocycle (virtual) super-rigidity result with discrete targets for p.m.p. profinite actions of property (T) groups. Here, we prove a similar result for property (τ) groups, but with some restrictions on the targets. Recall that a (residually finite) group Γ is said to *have the property* (τ) if the trivial representation is isolated among finite unitary representations. See [Lu, LZ] for more information on this property.

Theorem C. *Let $\Gamma = \Gamma_1 \times \Gamma_2$ be a group with the property (τ) and $\Gamma \curvearrowright X = \varprojlim X_n$ be a p.m.p. profinite action with growth condition such that both $\Gamma_i \curvearrowright X$ are ergodic. Let Λ be a residually-finite group. Then, any cocycle*

$$\alpha \colon \Gamma \times X \to \Lambda$$

virtually untwists, i.e., there exist $n \in \mathbb{N}$ and a cocycle $\beta \colon \Gamma \times X_n \to \Lambda$ which is equivalent to α.

It is plausible that the residual finiteness assumption on Λ is in fact redundant. Since there are groups having both properties $(\mathrm{HH})^+$ and (τ), Theorems A and C together imply a rigidity result for group measure space von Neumann algebras. Let $\Gamma' \leq \Gamma$ be a finite index subgroup and $\Gamma' \curvearrowright (X', \mu')$ be a m.p. action. Then, the induced action $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\Gamma' \curvearrowright X')$ is the Γ-action on the measure space $\Gamma/\Gamma' \times X'$, given by $g(p, x) = (gp, \sigma(gp)^{-1} g \sigma(p) x)$, where σ is a fixed cross section $\sigma \colon \Gamma/\Gamma' \to \Gamma$. (The action is unique up to conjugacy.) We say that two p.m.p. actions $\Gamma_i \curvearrowright (X_i, \mu_i)$, $i = 1, 2$, are *strongly virtually isomorphic* if there are a p.m.p. action $\Gamma' \curvearrowright (X', \mu')$ and finite index inclusions $\Gamma' \hookrightarrow \Gamma_i$ such that $\Gamma_i \curvearrowright X_i$ are measure-preservingly conjugate to $\mathrm{Ind}_{\Gamma'}^{\Gamma_i}(\Gamma' \curvearrowright X')$.

Corollary C. *Let $\Gamma_i = \mathrm{PSL}(2, \mathbb{Z}[\sqrt{2}])$ and $p_1 < p_2 < \cdots$ be prime numbers. Let $\Gamma = \Gamma_1 \times \Gamma_2$ act on $X = \varprojlim \mathrm{PSL}(2, (\mathbb{Z}/p_1 \cdots p_n \mathbb{Z})[\sqrt{2}])$ by the left-and-right translation action. Let $\Lambda \curvearrowright Y$ be any free ergodic p.m.p. action of a residually-finite group Λ and suppose that $L^\infty(X) \rtimes \Gamma \cong (L^\infty(Y) \rtimes \Lambda)^t$ for some $t > 0$. Then, $t \in \mathbb{Q}$ and the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are strongly virtually isomorphic.*

Connes and Jones ([CJ]) gave a first example of II_1 factors having more than one Cartan subalgebra. We present here a new class of examples. To describe it, recall first that if Γ is a discrete group having an infinite normal abelian subgroup H, then $L(H)$ is a Cartan subalgebra of $L(\Gamma)$ if and only if it satisfies the relative ICC condition: for any $g \in \Gamma \setminus H$, the set $\{aga^{-1} : a \in H\}$ is infinite. The group $H \rtimes \Gamma$ acts on H by $(a, g)b = agbg^{-1}$ (cf. Proposition 2.11 in [OP]).

Theorem D. *Let $\Gamma \curvearrowright X$ be a free ergodic p.m.p. action of a discrete group Γ having an infinite normal abelian subgroup H satisfying the relative ICC condition. Assume that $H \curvearrowright X$ is ergodic and profinite. Then, both $L^\infty(X)$ and $L(H)$ are Cartan subalgebras of $L^\infty(X) \rtimes \Gamma$. Assume moreover that $\Gamma \curvearrowright X$ is profinite and there is no $H \rtimes \Gamma$-invariant mean on $\ell^\infty(H)$. Then, the Cartan subalgebras $L^\infty(X)$ and $L(H)$ are non-conjugate.*

We distinguish two Cartan subalgebras by weak compactness. The simplest example is the following. Another example will be presented in Section 7.

Corollary D. *Let $p_1, p_2, \dots$ be prime numbers. Then the II_1-factor*

$$M = L^\infty(\varprojlim(\mathbb{Z}/p_1 \cdots p_n\mathbb{Z})^2) \rtimes (\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}))$$

has more than one Cartan subalgebra.

We observe that in the above, $L(\mathbb{Z}^2)$ is actually an (strong) HT Cartan subalgebra of M, in the sense of [Po1]. Thus, while an HT factor has unique HT Cartan subalgebra, up to unitary conjugacy, there exist HT factors that have at least two non-conjugate Cartan subalgebras. It is plausible that there is no essentially-free group action which gives rise to the same orbit equivalence relation as $(L(\mathbb{Z}^2) \subset M)$. Such examples were first exhibited by Furman ([Fu]). See also [MS] and [Po3].

2. Groups with the property (HH)

Let G be a locally compact group. We recall that a unitary Γ-representation $(\pi, \mathcal{H})$ is called *amenable* if there is a state φ on $\mathbb{B}(\mathcal{H})$ which is $\mathrm{Ad}\,\pi$-invariant: $\varphi \circ \mathrm{Ad}\,\pi_g = \varphi$ for all $g \in G$. This notion was introduced and studied by Bekka ([Be]). Among other things, he proved that π is amenable if and only if $\pi \otimes \bar{\pi}$ weakly contains the trivial representation.

Let σ be the conjugate action of G on $L^\infty(G)$: $(\sigma_h f)(g) = f(h^{-1}gh)$ for $f \in L^\infty(G)$ and $g, h \in G$. We say a locally compact group G is *inner-amenable* if there is a σ-invariant state μ on $L^\infty(G)$ which vanishes on $C_0(G)$. We note that in several literatures it is only required that μ is σ-invariant and $\mu \neq \delta_e$ (in case G is discrete).

Proposition 2.1. *A locally compact group G with the property* (HH) *has the Haagerup property and is not inner-amenable.*

Proof. Let $(b,\pi,\mathcal{K})$ be a proper 1-cocycle and suppose that there is a singular σ-invariant state μ on $L^\infty(G)$. For $x\in\mathbb{B}(\mathcal{H})$, we define $f_x\in L^\infty(G)$ by $f_x(g)=\|b(g)\|^{-2}\langle xb(g),b(g)\rangle$. Let $h\in G$ be fixed. Since

$$\|b(h^{-1}gh)-\pi_h^{-1}b(g)\|=\|b(h^{-1})+\pi_{h^{-1}g}b(h)\|\le 2\|b(h)\|,$$

and $\|b(g)\|\to\infty$ as $g\to\infty$, one has $\sigma_h(f_x)-f_{\pi_h x\pi_h^*}\in C_0(G)$. It follows that the state φ on $\mathbb{B}(\mathcal{H})$ defined by $\varphi(x)=\mu(f_x)$ is $\operatorname{Ad}\pi$-invariant. This means π is amenable. □

We do not know whether the converse is also true. Combined with Proposition 2.11 in [OP], the above proposition yields the following.

Corollary 2.2. *A discrete group Γ with the property* (HH)$^+$ *does not have an infinite normal amenable subgroup.*

We are indebted to Y. Shalom and Y. de Cornulier respectively for (3) and (4) of the next statement.

Theorem 2.3. *The following are true.*

(1) *Each of the properties* (HH), (HH)$^+$, *strong* (HH) *and strong* (HH)$^+$ *inherits to a lattice of a locally compact group.*
(2) *If G_1 and G_2 have the property* (HH) *(resp.* (HH)$^+$*), then so does $G_1\times G_2$.*
(3) *The groups* $\mathrm{SO}(n,1)$ *with $n\ge 2$ and* $\mathrm{SU}(n,1)$ *have the property* (HH)$^+$.
(4) *The groups* $\mathrm{SL}(2,\mathbb{R})$ *and* $\mathrm{SL}(2,\mathbb{C})$ *have the property strong* (HH)$^+$.
(5) *Suppose Γ is a countable non-amenable group acting properly on a finite-dimensional* CAT(0) *cube complex. If all hyperplane stabilizer groups are non-co-amenable, then Γ has the property* (HH)$^+$. *If all hyperplane stabilizer groups are amenable, then Γ has the property strong* (HH)$^+$.

Proof. The assertion (1) follows from the fact that the restriction of non-amenable (resp. weakly sub-regular) representation to a lattice is non-amenable (resp. weakly sub-regular). For the assertion (2), just consider the direct sum of 1-cocycles. We prove the property (HH)$^+$ for $G=\mathrm{SO}(n,1)$ $(n\ge 2)$ and $\mathrm{SU}(n,1)$. It follows from Theorem 3 in [BV] that every non-trivial irreducible representation of G is non-amenable. Since G does not have the property (T), by [Sh1], there is a non-trivial irreducible representation with an unbounded 1-cocycle. But, by [Sh2], every unbounded 1-cocycles of G is proper. Thus, G has the property (HH). Weak amenability is proved in [dCH, Cow].

The irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ which have non-trivial 1-cocycles are found in the principal series (see Example 3 in [Gu]) and hence are weakly equivalent to the regular representation.

If a group Γ acts properly on a CAT(0) cube complex Σ, then it has a proper 1-cocycle into the $\ell^2(H)$, where H is the set of hyperplanes in Σ. (See [NR].) The unitary representation on $\ell^2(H)$ is non-amenable (resp. weakly contained in the regular representation) if and only if all hyperplane stabilizer subgroups

are non-co-amenable (resp. amenable). Weak amenability for finite-dimensional CAT(0) cube complexes is proved in [GH, Mi]. □

Note that by a result of [CSV], the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$ acts properly on an infinite-dimensional CAT(0) cube complex with all hyperplane stabilizer subgroups amenable (being subgroups of $\bigoplus_{\mathbb{F}_2} \mathbb{Z}/2\mathbb{Z}$). It follows that $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$ has the property strong (HH), but not $(\mathrm{HH})^+$.

3. Miscellaneous Results

We use the same conventions and notations as in the previous paper ([OP]). Thus the symbol "Lim" will be used for a state on $\ell^\infty(\mathbb{N})$, or more generally on $\ell^\infty(I)$ with I directed, which extends the ordinary limit, and that the abbreviation "u.c.p." stands for "unital completely positive." We say a map is *normal* if it is ultraweakly continuous. Whenever a *finite* von Neumann algebra M is being considered, it comes equipped with a distinguished faithful normal tracial state, denoted by τ. Any group action on a finite von Neumann algebra is assumed to preserve the tracial state τ. If $M = P \rtimes \Gamma$ is a crossed product von Neumann algebra, then the tracial state τ on M is given by $\tau(au_g) = \delta_{g,e}\tau(a)$ for $a \in P$ and $g \in \Gamma$. A von Neumann subalgebra $P \subset M$ inherits the tracial state τ from M, and the unique τ-preserving conditional expectation from M onto P is denoted by E_P. We denote by $\mathcal{Z}(M)$ the center of M; by $\mathcal{U}(M)$ the group of unitary elements in M; and by

$$\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) : (\mathrm{Ad}\, u)(P) = P\}$$

the normalizer group of P in M, where $(\mathrm{Ad}\, u)(x) = uxu^*$. A von Neumann subalgebra $P \subset M$ is called *regular* if $\mathcal{N}_M(P)'' = M$. A regular maximal abelian von Neumann subalgebra $A \subset M$ is called a *Cartan subalgebra*. We note that if $\Gamma \curvearrowright X$ is a free ergodic p.m.p. action, then $A = L^\infty(X)$ is a Cartan subalgebra in the crossed product $L^\infty(X) \rtimes \Gamma$. (See [FM].)

We recall the definition of weak compactness.

Definition. Let (P, τ) be a finite von Neumann algebra, and $\Gamma \curvearrowright P$ be a τ-preserving action. The action is called *weakly compact* if there is a net $\eta_n \in L^2(P \mathbin{\bar{\otimes}} \bar{P})_+$ such that

(1) $\|\eta_n - (v \otimes \bar{v})\eta_n\|_2 \to 0$ for $v \in \mathcal{U}(P)$;
(2) $\|\eta_n - \mathrm{Ad}(u \otimes \bar{u})(\eta_n)\|_2 \to 0$ for $u \in \Gamma$;
(3) $\langle (a \otimes 1)\eta_n, \eta_n\rangle = \tau(a)$ for all $a \in P$.

(These conditions force P to be amenable.) A von Neumann subalgebra P of M is called *weakly compact* if the action $\mathcal{N}_M(P) \curvearrowright P$ is weakly compact.

It is proved in [OP, Proposition 3.4] that if $\Gamma \curvearrowright Q$ is weakly compact, then Q is weakly compact in the crossed product $Q \rtimes \Gamma$.

Theorem 3.1 (Theorem 3.5 in [OP])**.** *Let M be a finite von Neumann algebra with the complete metric approximation property. Then, every amenable von Neumann subalgebra P is weakly compact in M.*

Let $Q \subset M$ be finite von Neumann algebras. Then, the conditional expectation E_Q can be viewed as the orthogonal projection e_Q from $L^2(M)$ onto $L^2(Q) \subset L^2(M)$. It satisfies $e_Q x e_Q = E_Q(x)e_Q$ for every $x \in M$. The *basic construction* $\langle M, e_Q\rangle$ is the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and e_Q. We note that $\langle M, e_Q\rangle$ coincides with the commutant of the right Q-action in $\mathbb{B}(L^2(M))$. The conditional expectation E_Q extends on $\langle M, e_Q\rangle$ by the formula $E_Q(z)e_Q = e_Q z e_Q$ for $z \in \langle M, e_Q\rangle$. The basic construction $\langle M, e_Q\rangle$ comes together with the faithful normal semi-finite trace Tr such that $\mathrm{Tr}(xe_Qy) = \tau(xy)$. We denote

$$C^*(Me_QM) = \text{the norm-closed linear span of } \{xe_Qy : x, y \in M\},$$

which is an ultraweakly dense C*-subalgebra of $\langle M, e_Q\rangle$. Suppose that θ is a τ-preserving u.c.p. map on M such that $\theta|_Q = \mathrm{id}_Q$. Then, θ can be regarded as a contraction on $L^2(M)$ which commutes the left and right Q-actions on $L^2(M)$. In particular, $\theta \in \langle M, e_Q\rangle$. See Section 1.3 in [Po1] for more information on the basic construction.

The following is Theorem A.1 in [Po1] (see also Theorem 2.1 in [Po2]).

Theorem 3.2. *Let $P, Q \subset M$ be finite von Neumann subalgebras. Then, the following are equivalent.*

1. *There exists a non-zero projection $e \in P' \cap \langle M, e_Q\rangle$ such that* $\mathrm{Tr}(e) < \infty$.
2. *There exist non-zero projections $p \in P$ and $q \in Q$, a normal $*$-homomorphism $\theta\colon pPp \to qQq$ and a non-zero partial isometry $v \in M$ such that*

$$\forall x \in pPp \quad xv = v\theta(x)$$

and $v^*v \in \theta(pPp)' \cap qMq$, $vv^* \in p(P' \cap M)p$.

Definition. Let $P, Q \subset M$ be finite von Neumann algebras. Following [Po2], we say that P *embeds into Q inside M*, and write $P \preceq_M Q$, if any of the conditions in Theorem 3.2 holds.

Let $P \subset \mathcal{N}$ be von Neumann algebras. We say a state φ on $\mathcal{N}$ is P-central if $\varphi(u^*xu) = \varphi(x)$ for all $u \in \mathcal{U}(P)$ and $x \in \mathcal{N}$, or equivalently $\varphi(ax) = \varphi(xa)$ for all $a \in P$ and $x \in \mathcal{N}$.

Lemma 3.3. *Let $P, Q \subset M$ be a finite von Neumann algebras, and φ be a P-central state on $\langle M, e_Q\rangle$ whose restriction to M is normal. If $P \not\preceq_M Q$, then φ vanishes on $C^*(Me_QM)$.*

Proof. We assume $\varphi(C^*(Me_QM)) \neq \{0\}$ and prove $P \preceq_M Q$. Since M sits inside the multiplier of $C^*(Me_QM)$, there is an approximate unit f_n of $C^*(Me_QM)$ such that $\|[f_n, u]\| \to 0$ for every $u \in \mathcal{U}(M)$. (That is, (f_n) is a quasi-central

approximate unit for the ideal $C^*(Me_QM)$ in the C*-algebra $M+C^*(Me_QM)$.) We may assume that each f_n belongs to the linear span of $\{xe_Qy : x,y\in M\}$. We define positive linear functionals φ_n and ψ on $\langle M,e_Q\rangle$ by $\varphi_n(z)=\varphi(f_nzf_n)$ and $\psi(z)=\mathrm{Lim}\,\varphi_n(z)$ for $z\in\langle M,e_Q\rangle$. We note that ψ is non-zero and still P-central. We claim that ψ is normal. We observe that the net (φ_n) actually norm converges to ψ (w.r.t. Lim), since $\mathrm{Lim}\,\psi(f_n)=\mathrm{Lim}\,\varphi(f_n)=\|\psi\|$. Hence, it suffices to show that each φ_n is normal. Now let n be fixed and $f_n=\sum_{i=1}^k x_ie_Qy_i$. Then, for any $z\in\langle M,e_Q\rangle_+$, one has

$$f_n^*zf_n=\sum_{i,j=1}^k y_i^*E_Q(x_i^*zx_j)e_Qy_j\leq\sum_{i,j=1}^k y_i^*E_Q(x_i^*zx_j)y_j\in M$$

since $[E_Q(x_i^*zx_j)]_{i,j=1}^k$ is a positive element in $\mathbb{M}_k(Q)$ which commutes with $\mathrm{diag}(e_Q,\dots,e_Q)$. Hence, one has

$$\varphi_n(z)=\varphi(f_n^*zf_n)\leq(\varphi|_M)(\sum_{i,j=1}^k y_i^*E_Q(x_i^*zx_j)y_j).$$

This implies that φ_n is normal, and thus so is ψ. It follows that ψ can be regarded as a positive non-zero element in $P'\cap L^1\langle M,e_Q\rangle$ (see Section IX.2 in [Ta]). Taking a suitable spectral projection, we are done. $\square$

We recall that A.1 in [Po1] shows the following:

Lemma 3.4. *Let A and B be Cartan subalgebras of a type II_1-factor M. If $A\preceq_M B$, then there exists $u\in\mathcal{U}(M)$ such that $uAu^*=B$.*

Finally, we state some elementary lemmas about u.c.p. maps and positive linear functionals. We include proofs for the reader's convenience.

Lemma 3.5. *Let (M,τ) be a finite von Neumann algebra and θ be a τ-symmetric u.c.p. map on M. Then for every $a,x\in M$, one has*

$$\|\theta(ax)-\theta(a)\theta(x)\|_2\leq 2\|x\|_\infty\|a\|_\infty^{1/2}\|a-\theta(a)\|_2^{1/2}.$$

Proof. Let $\theta(x)=V^*\pi(x)V$ be a Stinespring dilation. Then,

$$\begin{aligned}\|\theta(ax)-\theta(a)\theta(x)\|_2&=\|V^*\pi(x^*)(1-VV^*)\pi(a^*)V\hat{1}\|_2\\&\leq\|x\|_\infty\|(1-VV^*)^{1/2}\pi(a^*)V\hat{1}\|_2\\&=\|x\|_\infty\tau\big(\theta(aa^*)-\theta(a)\theta(a^*)\big)^{1/2}.\end{aligned}$$

Since $\tau\circ\theta=\tau$, this completes the proof. $\square$

Lemma 3.6. *Let φ and ψ be positive linear functional on a C^*-algebra and $\varepsilon>0$. Suppose that $\varphi(1)\geq\psi(1)$ and $\varphi(x)-\psi(x)\leq\varepsilon\|x\|$ for all $x\geq 0$. Then, one has $\|\varphi-\psi\|\leq 2\varepsilon$.*

Proof. Let $\varphi - \psi = (\varphi - \psi)_+ - (\varphi - \psi)_-$ be the Hahn decomposition. Since $(\varphi - \psi)(1) \geq 0$, one has $\|(\varphi - \psi)_-\| \leq \|(\varphi - \psi)_+\| \leq \varepsilon$. □

4. Peterson's Deformation

We review the work of Peterson on real closable derivations, in order to give a qualitative version of Lemma 2.3 in [Pe].

Let (M, τ) be a finite von Neumann algebra. An M-M *bimodule* is a Hilbert space $\mathcal{H}$ together with normal representations λ of M and ρ of M^{op} such that $\lambda(M) \subset \rho(M^{\mathrm{op}})'$. The action of M is referred to as the left M-action and the action of M^{op} is referred to as the right M-action. We write intuitively $a\xi b$ for $\lambda(a)\rho(b^{\mathrm{op}})\xi$. By a *closable derivation*, we mean a map δ from a weakly dense $*$-subalgebra $\mathcal{D}$ of M into an M-M bimodule $\mathcal{H}$, which is closable as an operator from $L^2(M)$ into $\mathcal{H}$ and satisfies the Leibniz's rule:

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for every $x, y \in \mathcal{D}$. Moreover, a derivation is always assumed to be *real*: there is a conjugate-linear isometric involution J on $\mathcal{H}$ such that $J(x\delta(y)z) = z^*\delta(y^*)x^*$ for every $x, y, z \in \mathcal{D}$ (which is equivalent to another definition: $\langle \delta(x), \delta(y)z \rangle = \langle z^*\delta(y^*), \delta(x^*) \rangle$ for every $x, y, z \in \mathcal{D}$).

Let $\mathcal{H}$ be an M-M bimodule and $\delta\colon M \to \mathcal{H}$ be a closable derivation whose closure is denoted by $\bar{\delta}$. Thanks to the important work of [DL, Sa1], $\operatorname{dom} \bar{\delta} \cap M$ is still a weakly dense $*$-subalgebra and $\bar{\delta}$ satisfies the Leibniz's rule there. Hence, for notational simplicity, the closure $\bar{\delta}$ will be written as δ. We recycle some notations from [Pe]:

$$\Delta = \delta^*\delta,\ \zeta_\alpha = \sqrt{\frac{\alpha}{\alpha + \Delta}},\ \tilde{\delta}_\alpha = \alpha^{-1/2}\delta \circ \zeta_\alpha$$

(note that $\operatorname{ran} \zeta_\alpha \subset \operatorname{dom} \Delta^{1/2} = \operatorname{dom} \delta$) and

$$\tilde{\Delta}_\alpha = \alpha^{-1/2}\Delta^{1/2} \circ \zeta_\alpha = \sqrt{\frac{\Delta}{\alpha + \Delta}},\ \theta_\alpha = 1 - \tilde{\Delta}_\alpha.$$

All operators are firstly defined as Hilbert space operators. Since $1 - \sqrt{t} \leq \sqrt{1-t}$ for all $0 \leq t \leq 1$, one has $\theta_\alpha \leq \zeta_\alpha$ and

$$\|a - \zeta_\alpha(a)\|_2 \leq \|\tilde{\Delta}_\alpha(a)\|_2 = \|\tilde{\delta}_\alpha(a)\|_2 \leq \|a\|_2 \leq \|a\|_\infty$$

for all $a \in M$. By Lemma 2.2 in [Pe], the operators ζ_α and θ_α map $M \subset L^2(M)$ into M and are τ-symmetric u.c.p. on M.

We recall from [Sa2] the following facts: $\psi_t = \exp(-t\Delta^{1/2})$ form a semigroup of u.c.p. maps on M. Let

$$\begin{aligned}\Gamma(b^*, c) &= \Delta^{1/2}(b^*)c + b^*\Delta^{1/2}(c) - \Delta^{1/2}(b^*c)\\ &= \lim_{t\to 0} \frac{\psi_t(b^*c) - \psi_t(b^*)\psi_t(c)}{t} \in L^2(M)\end{aligned}$$

for $b, c \in \operatorname{dom}\Delta^{1/2} \cap M$ and note that

$$\langle \sum_i b_i \otimes y_i, \sum_j c_j \otimes z_j \rangle_\Gamma = \sum_{i,j} \tau(y_i^* \Gamma(b_i^*, c_j) z_j)$$

is a positive semi-definite form on $(\operatorname{dom}\Delta^{1/2} \cap M) \otimes M$. In particular, one has

$$|\tau(x^*\Gamma(b^*, c)y)| \leq \tau(x^*\Gamma(b^*, b)x)^{1/2}\tau(y^*\Gamma(c^*, c)y)^{1/2}.$$

It follows that

$$\begin{aligned}\|\Gamma(b^*, c)\|_2 &= \sup\{|\tau(x^*\Gamma(b^*, c)y)| : x, y \in M,\ \|xx^*\|_2 \leq 1, \|yy^*\|_2 \leq 1\} \\ &\leq \sup\{\tau(x^*\Gamma(b^*, b)x)^{1/2}\tau(y^*\Gamma(c^*, c)y)^{1/2} : \ \text{———} \ \} \\ &\leq \|\Gamma(b^*, b)\|_2^{1/2} \|\Gamma(c^*, c)\|_2^{1/2} \\ &\leq 4\|b\|_\infty^{1/2}\|\delta(b)\|^{1/2}\|c\|_\infty^{1/2}\|\delta(c)\|^{1/2}.\end{aligned}$$

Lemma 4.1 (Lemma 2.3 in [Pe])**.** *For every $a, x \in M$, one has*

$$\|\zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(ax)\| \leq 10\|x\|_\infty\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}$$

and

$$\|\tilde{\delta}_\alpha(x)\zeta_\alpha(a) - \tilde{\delta}_\alpha(xa)\| \leq 10\|x\|_\infty\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}.$$

Proof. One has

$$\zeta_\alpha(a)\tilde{\delta}_\alpha(x) = \alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x) =: A_1 - A_2.$$

We note that $\|A_2\| \leq \|x\|_\infty\|\tilde{\delta}_\alpha(a)\|$. Let $\delta = V\Delta^{1/2}$ be the polar decomposition. Then, one has

$$\begin{aligned}V^*A_1 &= \zeta_\alpha(a)\tilde{\Delta}_\alpha(x) + \tilde{\Delta}_\alpha(a)\zeta_\alpha(x) - \alpha^{-1/2}\Gamma(\zeta_\alpha(a), \zeta_\alpha(x)) \\ &=: \qquad B_1 \qquad + \qquad B_2 \qquad - \qquad B_3\end{aligned}$$

in $L^2(M)$. We note that $\|B_2\| \leq \|x\|_\infty\|\tilde{\delta}_\alpha(a)\|$; and by the estimate preceding to this lemma that $\|B_3\| \leq 4\|x\|_\infty\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}$. Finally, one has

$$B_1 = \zeta_\alpha(a)\tilde{\Delta}_\alpha(x) = \zeta_\alpha(a)(1 - \theta_\alpha)(x) \approx ax - \theta_\alpha(ax) = \tilde{\Delta}_\alpha(ax).$$

For the above estimates, we used

$$\|\zeta_\alpha(a)x - ax\|_2 \leq \|x\|_\infty\|a - \zeta_\alpha(a)\|_2 \leq \|x\|_\infty\|\tilde{\delta}_\alpha(a)\|_2$$

and

$$\begin{aligned}\|\zeta_\alpha(a)\theta_\alpha(x) - \theta_\alpha(ax)\|_2 &\leq \|x\|_\infty\|(\zeta_\alpha - \theta_\alpha)(a)\|_2 + \|\theta_\alpha(a)\theta_\alpha(x) - \theta_\alpha(ax)\|_2 \\ &\leq \|x\|_\infty(\|\tilde{\delta}_\alpha(a)\|_2 + 2\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2})\end{aligned}$$

(see Lemma 3.5). Consequently, one has

$$\zeta_\alpha(a)\tilde{\delta}_\alpha(x) \approx A_1 \approx VB_1 \approx \tilde{\delta}_\alpha(ax).$$

This yields the first inequality. Since the derivation is real, one obtains the second as well. □

We will need a vector-valued analogue of the above lemma. Let

$$\Omega = \{\eta \in L^2(M \bar{\otimes} \bar{M}) : (\mathrm{id} \otimes \bar{\tau})(\eta^*\eta) \leq 1 \text{ and } (\mathrm{id} \otimes \bar{\tau})(\eta\eta^*) \leq 1\}.$$

We note that if $\{\xi_k\}$ is an orthonormal basis of $L^2(M)$ and $\eta = \sum_{k=1}^{\infty} x_k \otimes \bar{\xi}_k$, then $(\mathrm{id} \otimes \bar{\tau})(\eta^*\eta) = \sum_k x_k^* x_k$ and $(\mathrm{id} \otimes \bar{\tau})(\eta\eta^*) = \sum_k x_k x_k^*$. (These series converge *a priori* in $L^1(M)$.) We note that if $\eta \in \Omega$ and $b, c \in M$ with $\|b\|_\infty \|c\|_\infty \leq 1$, then $\eta^*, (b \otimes 1)\eta(c \otimes 1) \in \Omega$.

Lemma 4.2. *For every $a \in M$ and $\eta \in \Omega$, one has*

$$\|(\zeta_\alpha(a) \otimes 1)(\tilde{\delta}_\alpha \otimes 1)(\eta) - (\tilde{\delta}_\alpha \otimes 1)((a \otimes 1)\eta)\|_{\mathcal{H}\bar{\otimes}L^2(\bar{M})} \leq 20\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}$$

and

$$\|(\tilde{\delta}_\alpha \otimes 1)(\eta)(\zeta_\alpha(a) \otimes 1) - (\tilde{\delta}_\alpha \otimes 1)(\eta(a \otimes 1))\|_{\mathcal{H}\bar{\otimes}L^2(\bar{M})} \leq 20\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}.$$

Proof. Let $a \in M$ be fixed and define a linear map $T \colon M \to \mathcal{H}$ by

$$T(x) = \zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(ax).$$

By Lemma 4.1, one has $\|T\| \leq 10\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}$. By the noncommutative little Grothendieck theorem (Theorem 9.4 in [Pi]), there are states f and g on M such that

$$\|T(x)\|^2 \leq \|T\|^2\big(f(x^*x) + g(xx^*)\big)$$

for all $x \in M$. It follows that for $\eta = \sum_{k=1}^{\infty} x_k \otimes \bar{\xi}_k \in \Omega$, one has

$$\begin{aligned}\|(\zeta_\alpha(a) \otimes 1)(\tilde{\delta}_\alpha \otimes 1)(\eta) - (\tilde{\delta}_\alpha \otimes 1)((a \otimes 1)\eta)\|^2_{\mathcal{H}\bar{\otimes}L^2(\bar{M})} \\ = \sum_k \|T(x_k)\|^2 \leq \sum_k \|T\|^2\big(f(x_k^*x_k) + g(x_kx_k^*)\big) \leq 2\|T\|^2.\end{aligned}$$

The second inequality follows similarly. □

5. Proof of Theorems A and B

Let Γ be a group and $(b, \pi, \mathcal{K})$ be a proper 1-cocycle. Replacing $(b, \pi, \mathcal{K})$ with $(b \oplus \bar{b}, \pi \oplus \bar{\pi}, \mathcal{K} \oplus \bar{\mathcal{K}})$ and considering an operator defined by $J_0(\xi \oplus \bar{\eta}) = \eta \oplus \bar{\xi}$ if necessary, we may assume that there is a conjugate-linear involution J_0 on $\mathcal{K}$ such that $J_0 b(g) = b(g)$ and $J_0\pi_g J_0 = \pi_g$ for all $g \in \Gamma$. (Note that π is amenable (resp. weakly sub-regular) if and only if so is $\pi \oplus \bar{\pi}$.)

Let $M = Q \rtimes \Gamma$ be the crossed product von Neumann algebra of a finite von Neumann algebra (Q, τ) by a τ-preserving action σ of Γ. We denote by u_g the

element in M that corresponds to $g \in \Gamma$. We equip $\mathcal{H} = L^2(Q) \otimes \ell^2(\Gamma) \otimes \mathcal{K}$ with an M-M bimodule structure by the following:

$$\begin{array}{rcccccc} \mathcal{H} & = & L^2(Q) & \otimes & \ell^2(\Gamma) & \otimes & \mathcal{K} \\ \text{left action by } g \in \Gamma & : & \sigma_g & \otimes & \lambda_g & \otimes & \pi_g \\ \text{left action by } a \in Q & : & a & \otimes & 1 & \otimes & 1 \\ \text{right action by } g \in \Gamma & : & 1 & \otimes & \rho_g^{-1} & \otimes & 1 \\ \text{right action by } a \in Q & : & \sum_{h\in\Gamma} \sigma_h(a)^{\mathrm{op}} & \otimes & e_h & \otimes & 1 \end{array}$$

We define a conjugate-linear involution J on $\mathcal{H}$ by

$$J(\widehat{a} \otimes \delta_g \otimes \xi) = -\widehat{\sigma_{g^{-1}}(a^*)} \otimes \delta_{g^{-1}} \otimes J_0\pi_{g^{-1}}\xi,$$

and the derivation $\delta\colon M \to \mathcal{H}$ by

$$\delta(au_g) = \widehat{a} \otimes \delta_g \otimes b(g) \in L^2(Q) \otimes \ell^2(\Gamma) \otimes \mathcal{K}$$

for $a \in Q$ and $g \in \Gamma$. It is routine to check that J intertwines the left and the right M-actions, $J\delta(au_g) = \delta(\sigma_{g^{-1}}(a^*)u_{g^{-1}}) = \delta((au_g)^*)$ and moreover that δ is a real closable derivation satisfying

$$\Delta(au_g) = \|b(g)\|^2 au_g,\ \zeta_\alpha(au_g) = \sqrt{\frac{\alpha}{\alpha + \|b(g)\|^2}}au_g$$

and

$$\theta_\alpha(au_g) = \left(1 - \sqrt{\frac{\|b(g)\|^2}{\alpha + \|b(g)\|^2}}\right) au_g$$

for all $a \in Q$ and $g \in \Gamma$. In particular, all θ_α belong to $C^*(Me_QM)$.

Lemma 5.1. *Suppose that π is weakly contained in the regular representation. Then, the M-M bimodule $\mathcal{H}$ is weakly contained in the coarse bimodule $L^2(M) \bar{\otimes} L^2(M)$. In particular, the left M-action on $\mathcal{H}$ extends to a u.c.p. map $\Psi\colon \mathbb{B}(L^2(M)) \to \mathbb{B}(\mathcal{H})$ whose range commutes with the right M-action.*

Proof. It is well-known and not hard to see that if π is weakly contained in the left regular representation λ, then the M-M bimodule $\mathcal{H}$ is weakly contained in $\hat{\mathcal{H}} := L^2(Q) \bar{\otimes} \ell^2(\Gamma) \bar{\otimes} \ell^2(\Gamma)$, where $(\pi, \mathcal{K})$ is replaced with $(\lambda, \ell^2(\Gamma))$ in the definition of $\mathcal{H}$. Let U be the unitary operator on $\hat{\mathcal{H}}$ defined by

$$U\widehat{a} \otimes \delta_h \otimes \delta_g = \widehat{\sigma_g(a)} \otimes \delta_{gh} \otimes \delta_g.$$

It is routine to check that $U^*\lambda(M)U \subset \lambda(Q) \bar{\otimes} \mathbb{C}1 \bar{\otimes} \mathbb{B}(\ell^2(\Gamma))$ and $U^*\rho(M^{\mathrm{op}})U \subset \rho(Q^{\mathrm{op}}) \bar{\otimes} \mathbb{B}(\ell^2(\Gamma)) \bar{\otimes} \mathbb{C}1$, where λ and ρ respectively stand for the left and right actions on $\hat{\mathcal{H}}$. Since the ambient von Neumann algebras are amenable and commuting, $\hat{\mathcal{H}}$ and *a fortiori* $\mathcal{H}$ is weakly contained in the coarse M-M bimodule, i.e., the binormal representation μ of $M \otimes M^{\mathrm{op}}$ on $\mathcal{H}$ is continuous w.r.t. the minimal tensor norm. Hence, μ extends to a u.c.p. map $\tilde{\mu}$ from $\mathbb{B}(L^2(M)) \bar{\otimes} M^{\mathrm{op}}$

into $\mathbb{B}(\mathcal{H})$. We define $\Psi\colon \mathbb{B}(L^2(M)) \to \mathbb{B}(\mathcal{H})$ by $\Psi(x) = \tilde{\mu}(x \otimes 1)$. Since M^{op} is in the multiplicative domain of $\tilde{\mu}$, the range of Ψ commutes with the right M-action. $\square$

For the following, let $P \subset M$ be an amenable von Neumann subalgebra such that $P \not\preceq_M Q$, and $\mathcal{G} \subset \mathcal{N}_M(P)$ be a subgroup whose action on P is weakly compact. We may and will assume that $\mathcal{U}(P) \subset \mathcal{G}$. By definition, there exists a sequence $\eta_n \in L^2(M \bar{\otimes} \bar{M})_+$ such that

(1) $\|\eta_n - (v \otimes \bar{v})\eta_n\|_2 \to 0$ for $v \in \mathcal{U}(P)$;
(2) $\|\eta_n - \mathrm{Ad}(u \otimes \bar{u})(\eta_n)\|_2 \to 0$ for $u \in \mathcal{G}$;
(3) $\langle (a \otimes 1)\eta_n, \eta_n\rangle = \tau(a)$ for all $a \in M$.

We note that $\eta_n \in \Omega$.

Lemma 5.2. *For every $\alpha > 0$ and $a \in M$, one has*

$$\operatorname*{Lim}_n \|(\tilde{\delta}_\alpha \otimes 1)((a \otimes 1)\eta_n)\| = \|a\|_2.$$

Proof. Note that $\|(a \otimes 1)\eta_n\|_2 = \|a\|_2$. Define a state on $\langle M, e_Q\rangle$ by

$$\varphi_0(x) = \operatorname*{Lim}_n \langle (x \otimes 1)\eta_n, \eta_n\rangle.$$

By construction, φ_0 is a P-central state such that $\varphi_0|_M = \tau$. Since $P \not\preceq_M Q$ and $\theta_\alpha a \in C^*(Me_QM)$, Lemma 3.5 implies $\varphi_0(a^*\theta_\alpha^*\theta_\alpha a) = 0$. It follows that

$$\begin{aligned}\operatorname*{Lim}_n \|(\tilde{\delta}_\alpha \otimes 1)((a \otimes 1)\eta_n)\| &= \operatorname*{Lim}_n \|((1-\theta_\alpha)a \otimes 1)\eta_n\|_2\\ &= \operatorname*{Lim}_n \|(a \otimes 1)\eta_n\|_2\\ &= \|a\|_2.\end{aligned}$$

This completes the proof. $\square$

For $\alpha > 0$, a non-zero projection $p \in \mathcal{G}' \cap M$ and n, we denote

$$\eta_n^{p,\alpha} = (\tilde{\delta}_\alpha \otimes 1)((p \otimes 1)\eta_n)$$

and define a state $\varphi_{p,\alpha}$ on $\mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$, where $\rho(M^{\mathrm{op}})$ is the right M-action on $\mathcal{H}$, by

$$\varphi_{p,\alpha}(x) = \|p\|_2^{-2} \operatorname*{Lim}_n \langle (x \otimes 1)\eta_n^{p,\alpha}, \eta_n^{p,\alpha}\rangle.$$

Lemma 5.3. *Let $a \in \mathcal{G}''$. Then, one has*

$$\operatorname*{Lim}_\alpha |\varphi_{p,\alpha}(\zeta_\alpha(a)x) - \varphi_{p,\alpha}(x\zeta_\alpha(a))| = 0$$

uniformly for $x \in \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$ with $\|x\|_\infty \le 1$.

Proof. Let $u \in \mathcal{G}$ and denote $u_\alpha = \zeta_\alpha(u)$. By Lemma 4.2, one has

$$\operatorname*{Lim}_n \|\eta_n^{p,\alpha} - (u_\alpha \otimes \bar{u})\eta_n^{p,\alpha}(u_\alpha \otimes \bar{u})^*\| \le 40\|\tilde{\delta}_\alpha(u)\|^{1/2}.$$

Since $u_\alpha^* u_\alpha \le 1$, one has for every $x \in (\rho(M^{\mathrm{op}})')_+$ that

$$\begin{aligned}\varphi_{p,\alpha}(u_\alpha^* x u_\alpha) &\ge \|p\|_2^{-2} \operatorname*{Lim}_n \langle (x\otimes 1)(u_\alpha \otimes \bar{u})\eta_n^{p,\alpha}(u_\alpha\otimes\bar{u})^*, (u_\alpha\otimes\bar{u})\eta_n^{p,\alpha}(u_\alpha\otimes\bar{u})^*\rangle \\ &\ge \varphi_{p,\alpha}(x) - 80\|p\|_2^{-2}\|\tilde{\delta}_\alpha(u)\|^{1/2}\|x\|_\infty.\end{aligned}$$

By Lemma 3.6, one obtains

$$\|\varphi_{p,\alpha}(\,\cdot\,) - \varphi_{p,\alpha}(u_\alpha^* \cdot u_\alpha)\| \le 160\|p\|_2^{-2}\|\tilde{\delta}_\alpha(u)\|^{1/2}.$$

In particular, $\operatorname{Lim}_\alpha \varphi_{p,\alpha}(1-u_\alpha^* u_\alpha) = 0$ and

$$\operatorname*{Lim}_\alpha |\varphi_{p,\alpha}(u_\alpha x) - \varphi_{p,\alpha}(x u_\alpha)| = 0$$

uniformly for x with $\|x\|_\infty \le 1$. This implies that

$$\operatorname*{Lim}_\alpha |\varphi_{p,\alpha}(\zeta_\alpha(a)x) - \varphi_{p,\alpha}(x\zeta_\alpha(a))| = 0$$

for each $a \in \operatorname{span}\mathcal{G}$ and uniformly for $x \in \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$ with $\|x\|_\infty \le 1$. However, by Lemma 4.2,

$$\begin{aligned}|\varphi_{p,\alpha}(x\zeta_\alpha(a))| &= \|p\|_2^{-2}|\operatorname*{Lim}_n \langle (x\otimes 1)(\zeta_\alpha(a)\otimes 1)\eta_n^{p,\alpha}, \eta_n^{p,\alpha}\rangle| \\ &\le \|p\|_2^{-1}\|x\|_\infty\big(20\|a\|_\infty^{1/2}\|a\|_2^{1/2} + \|a\|_2\big),\end{aligned}$$

and likewise for $|\varphi_{p,\alpha}(\zeta_\alpha(a)x)|$. Thus, by Kaplansky's Density Theorem, we are done. $\square$

Now, we are in position to prove Theorems A and B.

Proof of Theorem A. Let $\mathcal{G}'' = M$ and $\varphi_\alpha = \varphi_{1,\alpha}$. By Lemma 5.3, one has

$$\operatorname*{Lim}_\alpha |\varphi_\alpha(\zeta_\alpha(a)x) - \varphi_\alpha(x\zeta_\alpha(a))| = 0$$

for every $a \in \mathcal{G}'' = M$ and $x \in \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$. Since

$$\|u_g - \zeta_\alpha(u_g)\| = 1 - \sqrt{\frac{\alpha}{\alpha + \|b(g)\|^2}} \to 0$$

as $\alpha \to \infty$, one has

$$\operatorname*{Lim}_\alpha |\varphi_\alpha(u_g x u_g^*) - \varphi_\alpha(x)| = 0$$

for every $g \in \Gamma$ and $x \in \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$. Hence, the state φ defined by

$$\varphi(x) = \operatorname*{Lim}_\alpha \varphi_\alpha(x)$$

on $\mathbb{B}(\mathcal{K}) \subset \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$ is $\operatorname{Ad}\pi$-invariant. Therefore, π is an amenable representation, in contradiction to the property (HH). $\square$

Proof of Theorem B. We use Haagerup's criterion for amenable von Neumann algebras (Lemma 2.2 in [Ha]). Let a non-zero projection $p \in \mathcal{G}' \cap M$ and a finite subset $F \subset \mathcal{U}(\mathcal{G}'')$ be given arbitrary. We need to show

$$\|\sum_{u \in F} up \otimes \overline{up}\|_{M \bar{\otimes} \bar{M}} = |F|.$$

Let $u \in \mathcal{U}(\mathcal{G}'')$. By Lemmas 4.2 and 5.2, one has

$$\begin{aligned}\varphi_{p,\alpha}(\zeta_\alpha(up)^*\zeta_\alpha(up)) &= \|p\|_2^{-2} \operatorname*{Lim}_n \|(\zeta_\alpha(up) \otimes 1)(\tilde{\delta}_\alpha \otimes 1)((p \otimes 1)\eta_n)\|_2^2 \\ &\geq \|p\|_2^{-2} \operatorname*{Lim}_n \big(\|(\tilde{\delta}_\alpha \otimes 1)((up \otimes 1)\eta_n)\|_2 - 20\|\tilde{\delta}_\alpha(up)\|^{1/2}\big)^2 \\ &\geq 1 - 40\|p\|_2^{-1}\|\tilde{\delta}_\alpha(up)\|^{1/2}.\end{aligned}$$

Hence, by Lemma 5.3, one has

$$\operatorname*{Lim}_\alpha |\varphi_{p,\alpha}(\zeta_\alpha(up)^* x \zeta_\alpha(up)) - \varphi_{p,\alpha}(x)| = 0$$

uniformly for $x \in \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$ with $\|x\|_\infty \leq 1$. By Lemma 5.1, the left M-action on $\mathcal{H}$ extends to a u.c.p. map $\Psi\colon \mathbb{B}(L^2(M)) \to \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$. The state $\psi_{p,\alpha} = \varphi_{p,\alpha} \circ \Psi$ on $\mathbb{B}(L^2(M))$ satisfies

$$\operatorname*{Lim}_\alpha |\psi_{p,\alpha}(\zeta_\alpha(up)^* x \zeta_\alpha(up)) - \psi_{p,\alpha}(x)| = 0$$

uniformly for $x \in \mathbb{B}(L^2(M))$ with $\|x\|_\infty \leq 1$. By a standard convexity argument in cooperation with the Powers-Størmer inequality, this implies that

$$\operatorname*{Lim}_\alpha \|\sum_{u \in F} \zeta_\alpha(up) \otimes \overline{\zeta_\alpha(up)}\|_{M \bar{\otimes} \bar{M}} = |F|$$

for the finite subset $F \subset \mathcal{U}(\mathcal{G}'')$. Since ζ_α are u.c.p. maps, this yields

$$\|\sum_{u \in F} up \otimes \overline{up}\|_{M \bar{\otimes} \bar{M}} \geq \operatorname*{Lim}_\alpha \|\sum_{u \in F} \zeta_\alpha(up) \otimes \overline{\zeta_\alpha(up)}\|_{M \bar{\otimes} \bar{M}} = |F|.$$

This completes the proof. □

Proof of Corollaries A and B. The corollaries follow from the corresponding Theorems and Lemma 3.4, because all the von Neumann algebras in consideration have the complete metric approximation property and hence all amenable subalgebras are weakly compact (Theorem 3.1). □

Remark 5.4. With the same argument as above one can show the following. Let (M, τ) be a finite von Neumann algebra having a real closable derivation $(\delta, \mathcal{H})$ such that (1) $\delta^*\delta$ has compact resolvent; and (2) $\mathcal{H}$ is weakly contained in the coarse bimodule. Then, for every diffuse amenable von Neumann subalgebra $P \subset M$ and any subgroup $\mathcal{G} \subset \mathcal{N}_M(P)$ whose action on P is weakly compact, the von Neumann subalgebra $\mathcal{G}''$ is amenable.

6. Cocycle rigidity

We fix a notation for profinite actions. An action $\Gamma \curvearrowright (X,\mu)$ is said to be *profinite* if (X,μ) is the projective limit of finite-cardinality probability spaces (X_n,μ_n) on which Γ acts consistently. We will identify $L^\infty(X_n,\mu_n)$ as the corresponding Γ-invariant finite-dimensional von Neumann subalgebra of $L^\infty(X,\mu)$. The same thing for L^2. We write $X=\bigsqcup_a X_{a,n}$ for the partition of X corresponding to X_n, i.e., the characteristic functions of $X_{a,n}$'s are the non-zero minimal projections in $L^\infty(X_n)$.

Definition. Let $\pi\colon \Gamma \curvearrowright \mathcal{H}$ be a unitary representation. We say π *has a spectral gap* if there are a finite subset $F\subset\Gamma$ and $\kappa>0$, called *a critical pair*, satisfying the following property: denoting by P the orthogonal projection of $\mathcal{H}$ onto the subspace of π-invariant vectors, one has

$$\kappa\|\xi-P\xi\|\le\max_{g\in F}\|\xi-\pi_g\xi\|$$

for every $\xi\in\mathcal{H}$. (This is equivalent to that the point 1 is isolated (if it exists) in the spectrum of the self-adjoint operator $(2|F|)^{-1}\sum_{g\in F}(\pi_g+\pi_g^*)$ on $\mathcal{H}$.) We say that π *has a stable spectral gap* if the unitary representation $\pi\otimes\bar\pi$ of Γ on $\mathcal{H}\bar\otimes\bar{\mathcal{H}}$ has a spectral gap. (Note that we allow $\operatorname{rank}P\ge1$.)

When the unitary representation π arises from a p.m.p. action $\Gamma\curvearrowright X$, we simply say $\Gamma\curvearrowright X$ *has a (stable) spectral gap* if π has. Assume moreover that the action $\Gamma\curvearrowright X$ is profinite. We say $\Gamma\curvearrowright X$ *has a stable spectral gap with growth condition* if there are a critical pair (F,κ) such that π^F, the restriction of π to the subgroup of Γ generated by F, does not have a subrepresentation of infinite multiplicity.

Suppose that $\Gamma\curvearrowright\varprojlim X_n$ has a stable spectral gap. Then, π^F has finitely many equivalence classes of irreducible subrepresentations of any given dimension $k\in\mathbb{N}$. (See [HRV].) It follows that the growth condition is equivalent to that the minimal dimension k_n of a non-zero subrepresentation of $\pi^F|_{L^2(X)\ominus L^2(X_n)}$ tends to infinity.

Lemma 6.1. *Let $\Gamma\curvearrowright X$ be a p.m.p. action which is profinite and has a stable spectral gap with growth condition. Let $F\subset\Gamma$ and $\kappa>0$ be a critical pair. Then, for any $k\in\mathbb{N}$ and unitary elements $\{u_g\}_{g\in F}$ on the k-dimensional Hilbert space ℓ_k^2, one has*

$$\frac{\kappa^2}{2}(1-\frac{k}{k_n})\|\xi-P_{L^2(X_n)\bar\otimes\ell_k^2}\xi\|_2\le\max_{g\in F}\|\xi-(\pi_g\otimes u_g)\xi\|_2$$

for every $\xi\in L^2(X)\bar\otimes\ell_k^2$ and $n\in\mathbb{N}$.

Proof. We denote $L^2(X_n)^\perp=L^2(X)\ominus L^2(X_n)$. It suffices to show

$$\frac{\kappa^2}{2}(1-\frac{k}{k_n})\|\xi\|_2\le\max_{g\in F}\|\xi-(\pi_g\otimes u_g)\xi\|_2$$

for $\xi \in L^2(X_n)^\perp \bar{\otimes} \ell^2_k$. We assume $\|\xi\|_2 = 1$ and denote the right hand side of the asserted inequality by ε. We view ξ as a Hilbert-Schmidt operator T_ξ from $\overline{\ell^2_k}$ into $L^2(X_n)^\perp$. Note that

$$\|T_\xi - \pi_g T_\xi \bar{u}_g^*\|_2 = \|\xi - (\pi_g \otimes u_g)\xi\|_2 \le \varepsilon.$$

Hence by the Powers-Størmer inequality, the Hilbert-Schmidt operator $S_\xi = (T_\xi T_\xi^*)^{1/2}$ on $L^2(X_n)^\perp$ satisfies

$$\begin{aligned}\|S_\xi - \pi_g S_\xi \pi_g^*\|_2^2 &\le \|T_\xi T_\xi^* - \pi_g T_\xi T_\xi^* \pi_g^*\|_1 \\ &\le \|T_\xi + \pi_g T_\xi \bar{u}_g^*\|_2 \|T_\xi - \pi_g T_\xi \bar{u}_g^*\|_2 \\ &\le 2\varepsilon.\end{aligned}$$

By the stable spectral gap property, one has

$$\|S_\xi - P(S_\xi)\|_2^2 \le 2\varepsilon/\kappa^2.$$

Since $P(S_\xi)$ commutes with π_g for all $g \in F$, growth condition implies that $P(S_\xi) = \sum_i \gamma_i r_i^{-1/2} Q_i$ for some $\gamma_i \ge 0$ and mutually orthogonal projections Q_i with $r_i = \mathrm{Tr}(Q_i) \ge k_n$. Since S_ξ has rank at most k, denoting its range projection by R, one has

$$\begin{aligned}\|P(S_\xi)\|_2^2 &= \langle S_\xi, P(S_\xi)\rangle \\ &= \sum_i \gamma_i r_i^{-1/2} \mathrm{Tr}(Q_i R S_\xi Q_i) \\ &\le \Big(\sum_i \gamma_i^2 r_i^{-1} \mathrm{Tr}(Q_i R Q_i)\Big)^{1/2} \Big(\sum_i \mathrm{Tr}(Q_i S_\xi^* S_\xi Q_i)\Big)^{1/2} \\ &\le \Big(\sum_i \gamma_i^2 k_n^{-1} k\Big)^{1/2} \|S_\xi\|_2 \\ &= (k/k_n)^{1/2} \|P(S_\xi)\|_2.\end{aligned}$$

By combining two inequalities, one obtains

$$1 - (k/k_n) \le 1 - \|P(S_\xi)\|_2^2 = \|S_\xi - P(S_\xi)\|_2^2 \le 2\varepsilon/\kappa^2$$

and hence the desired inequality. □

Recall that a cocycle of $\Gamma \curvearrowright X$ with values in a group Λ is a measurable map $\alpha\colon \Gamma \times X \to \Lambda$ satisfying the cocycle identity:

$$\forall g, h \in \Gamma,\ \mu\text{-a.e. } x \in X, \quad \alpha(g, hx)\alpha(h, x) = \alpha(gh, x).$$

A cocycle α which is independent of the x-variable is said to be *homomorphism* for the obvious reason. Cocycles α and β are said to be *equivalent* if there is a measurable map $\phi\colon X \to \Lambda$ such that $\beta(g,x) = \phi(gx)\alpha(g,x)\phi(x)^{-1}$ for all $g \in \Gamma$ and μ-a.e. $x \in X$.

Lemma 6.2. *Let $\Gamma = \Gamma_1 \times \Gamma_2$ and $\Gamma \curvearrowright X = \varprojlim X_n$ be a p.m.p. profinite action such that $\Gamma_2 \curvearrowright X$ has a stable spectral gap with growth condition. Let (N, τ) be a finite type* I *von Neumann algebra, and $\alpha\colon \Gamma \times X \to \mathcal{U}(N)$ be a cocycle. Then, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that*

$$\int_X \|\alpha(g,x) - \alpha'_{a(x),n}(g)\|_2^2 \, dx \le \varepsilon$$

for all $g \in \ker(\Gamma_1 \to \mathrm{Aut}(X_n))$, where $\alpha'_{a,n}(g) = |X_{a,n}|^{-1} \int_{X_{a,n}} \alpha(g,y)\, dy$ and $a(x)$ is such that $x \in X_{a(x),n}$.

Proof. It suffices to consider each direct summand of N and hence we may assume $N = \mathbb{M}_k(\mathbb{C}) \otimes A$, where A is an abelian von Neumann algebra. For every $g \in \Gamma$, we define $w_g \in L^\infty(X) \mathbin{\bar{\otimes}} N = L^\infty(X, N)$ by $w_g(x) = \alpha(g, g^{-1}x)$. Then, it becomes a unitary 1-cocycle for $\tilde{\sigma} = \sigma \otimes \mathrm{id}_N$:

$$w_{gh} = w_g \tilde{\sigma}_g(w_h).$$

Let $F \subset \Gamma_2$ and $\kappa > 0$ be a critical pair for the stable spectral gap of $\Gamma_2 \curvearrowright X$. Let $\delta = \varepsilon\kappa^2/8$ and take $m \in \mathbb{N}$ and unitary elements $w'_h \in L^\infty(X_m) \mathbin{\bar{\otimes}} N$ such that $\|w_h - w'_h\|_2 < \delta$ for every $h \in F$. For the rest of the proof, we fix $g \in \ker(\Gamma_1 \to \mathrm{Aut}(X_m))$. Since $w'_h = \tilde{\sigma}_g(w'_h)$, one has

$$w_g w'_h \approx w_g \tilde{\sigma}_g(w_h) = w_{gh} = w_{hg} \approx w'_h \tilde{\sigma}_h(w_g),$$

for every $h \in F$. We define trace-preserving $*$-automorphisms π_h on $L^\infty(X) \mathbin{\bar{\otimes}} N$ by

$$\pi_h(x) = \mathrm{Ad}(w'_h) \circ \tilde{\sigma}_h(x)$$

and note that $\|w_g - \pi_h(w_g)\|_2 \le 2\delta$ for every $h \in F$. We write $\tilde{\pi}_h$ for the restriction of π_h to $L^\infty(X_m) \mathbin{\bar{\otimes}} N$. Note that $\tilde{\pi}_h$ acts as identity on $\mathbb{C}1 \mathbin{\bar{\otimes}} A \subset L^\infty(X_m) \mathbin{\bar{\otimes}} N$.

Let $\{p_a\}$ be the set of non-zero minimal projections in $L^\infty(X_m)$ and define an isometry $V\colon L^2(X) \to L^2(X) \mathbin{\bar{\otimes}} L^2(X_m)$ by $V\xi = |X_m|^{1/2} \sum_a p_a\xi \otimes p_a$. (Here $|X_m|$ stands for the cardinality of the atoms of X_m.) We claim that $(V \otimes 1)\pi_h = (\sigma_h \otimes \tilde{\pi}_h)(V \otimes 1)$. Indeed, if $w'_h = \sum_a \sigma_h(p_a) \otimes y_a$, then

$$\begin{aligned}(\sigma_h \otimes \tilde{\pi}_h)(V \otimes 1)(\xi \otimes c) &= |X_m|^{1/2} (\sigma_h \otimes \tilde{\pi}_h) \sum_a p_a\xi \otimes p_a \otimes c \\ &= |X_m|^{1/2} \sum_a \sigma_h(p_a\xi) \otimes \sigma_h(p_a) \otimes y_a c y_a^* \\ &= V \sum_a \sigma_h(p_a\xi) \otimes y_a c y_a^* \\ &= V\pi_h(\xi \otimes c)\end{aligned}$$

for all $\xi \in L^2(X)$ and $c \in L^2(N)$. Now, it follows that

$$\max_{h \in F} \|(V \otimes 1)w_g - (\sigma_h \otimes \tilde{\pi}_h)(V \otimes 1)w_g\|_2 \le 2\delta.$$

We observe that if $\tilde{\pi}_h$ is viewed as a unitary operator on $L^2(X_m)\mathbin{\bar{\otimes}} L^2(\mathbb{M}_k(\mathbb{C}))\mathbin{\bar{\otimes}} L^2(A)$, then it lives in $\mathbb{B}(L^2(X_m)\mathbin{\bar{\otimes}} L^2(\mathbb{M}_k(\mathbb{C})))\mathbin{\bar{\otimes}} A$. Hence Lemma 6.1 applies and one obtains

$$\frac{\kappa^2}{2}(1-\frac{mk^2}{k_n})\|(V\otimes 1)w_g-(P_{L^2(X_n)}\otimes 1\otimes 1)(V\otimes 1)w_g\|_2\leq 2\delta$$

for every $n\in\mathbb{N}$. Finally take n to be such that $n\geq m$ and $k_n\geq 2mk^2$. Since $(P_{L^2(X_n)}\otimes 1)V=VP_{L^2(X_n)}$ for $n\geq m$, one has

$$\begin{aligned}(\int_X \|\alpha(g,x)-\alpha'_{a(x),n}(g)\|_2^2\,dx)^{1/2} &= \|w_g-(P_{L^2(X_n)}\otimes 1)w_g\|_2\\ &\leq 4\delta/(\kappa^2(1-(\frac{mk^2}{k_n})^{1/2}))\leq\varepsilon.\end{aligned}$$

We note that $\ker(\Gamma_1\to\mathrm{Aut}(X_n))\subset\ker(\Gamma_1\to\mathrm{Aut}(X_m))$. □

We combine the above result with results of Ioana in [Io], to obtain the following cocycle rigidity result for profinite actions of product groups.

Theorem 6.3. *Let $\Gamma=\Gamma_1\times\Gamma_2$ and $\Gamma\curvearrowright X$ be an ergodic p.m.p. profinite action such that $\Gamma_i\curvearrowright X$ has a stable spectral gap with growth condition, for each $i=1,2$. Let Λ be a finite group and $\alpha\colon\Gamma\times X\to\Lambda$ be a cocycle. Then, there exists a finite index subgroup $\Gamma'\subset\Gamma$ such that for each Γ'-ergodic component $X'\subset X$, the restricted cocycle $\alpha|_{\Gamma'\times X'}$ is equivalent to a homomorphism from Γ' into Λ.*

Proof. The proof of this theorem is very similar to that of Theorem B in [Io], and hence it will be rather sketchy. Let $Z=X\times X\times\Lambda$ and we will consider the unitary representation $\pi\colon\Gamma\curvearrowright L^2(Z)$ induced by the m.p. transformation

$$g(x,y,t)=(gx,gy,\alpha(g,x)t\alpha(g,y)^{-1}).$$

Let $\varepsilon>0$ be arbitrary. Since Λ is discrete, Lemma 6.2 implies that there are a normal finite index subgroup Γ' and $n\in\mathbb{N}$ such that $\|\pi(g)\xi_n-\xi_n\|_2<\varepsilon$ for all $g\in\Gamma'$, where $\xi_n=|X_n|^{1/2}\sum_a\chi_{X_{a,n}\times X_{a,n}\times\{e\}}$. It follows that the circumcenter of $\pi(\Gamma')\xi_n$ is a $\pi(\Gamma')$-invariant vector which is close to ξ_n. Since $\Gamma\curvearrowright X$ is ergodic and Γ' is a normal finite index subgroup in Γ, there are a Γ'-ergodic component $X'\subset X$ and a finite subset $E\subset\Gamma$ such that $X=\bigsqcup_{s\in E}sX'$. Thus, there are Γ'-ergodic components $X_1',X_2'\subset X$ such that $\xi'=|X'|^{-1}\chi_{X_1'\times X_2'\times\{e\}}$ is close to a $\pi(\Gamma')$-invariant vector. We may assume that $X_1'=X'$. By Corollary 2.2 in [Io], the cocycle $\alpha|_{\Gamma'\times X'}$ is equivalent to a homomorphism θ via $\phi\colon X'\to\Lambda$, i.e., $\theta(g)=\phi(gx)\alpha(g,x)\phi(x)^{-1}$. We observe that $\alpha|_{\Gamma'\times sX'}$ is equivalent to $\theta\circ\mathrm{Ad}(s^{-1})$.

Indeed, one has

$$\begin{aligned}\theta(s^{-1}gs) &= \phi(s^{-1}gsx)\alpha(s^{-1}gs, x)\phi(x)^{-1}\\ &= \phi(s^{-1}gsx)\alpha(s^{-1}, gsx)\alpha(g, sx)\alpha(s, x)\phi(x)^{-1}\\ &= \psi(gsx)\alpha(g, sx)\psi(sx)^{-1},\end{aligned}$$

where $\psi(sx) = \phi(x)\alpha(s^{-1}, sx)$ for $s \in E$ and $x \in X'$. □

Proof of Theorem C. By Theorem B and Remark 3.1 in [Io], it suffices to show that the unitary representation $\pi\colon \Gamma \curvearrowright L^2(X \times X \times \Lambda)$ has a spectral gap. Let Λ_j be the finite quotients of Λ. Since Λ is residually finite the unitary representation π is weakly contained in the direct sum $\bigoplus \pi_j$, where π_j is the unitary representation induced by $\Gamma \curvearrowright X \times X \times \Lambda_j$ using the same 1-cocycle composed with the quotient $\Lambda \to \Lambda_j$. Thus, it suffices to show that π_j's have a uniform spectral gap. We prove this by showing that each π_j is contained in a direct sum of finite representations; then the uniformity follows from property (τ). We may assume that Λ is finite and $\pi_j = \pi$. By Theorem 6.3, there is a finite index subgroup Γ' such that for each Γ'-ergodic component $X'_k \subset X$, the restricted cocycle $\alpha|_{\Gamma' \times X'_k}$ is equivalent to a homomorphism $\theta_k\colon \Gamma' \to \Lambda$ via $\phi_k\colon X'_k \to \Lambda$, i.e., $\theta_k(g) = \phi_k(gx)\alpha(g, x)\phi_k(x)^{-1}$. Let $\sigma_{k,l}$ be the automorphism on $X'_k \times X'_l \times \Lambda$ defined by $\sigma_{k,l}(x, y, t) = (x, y, \phi_k(x)t\phi_l(y)^{-1})$. Then, for $g \in \Gamma'$, one has

$$\begin{aligned}\sigma_{k,l}g\sigma_{k,l}^{-1}(x, y, t) &= (gx, gy, \phi_k(gx)\alpha(g, x)\phi_k(x)^{-1}t\phi_l(y)\alpha(g, y)^{-1}\phi_l(gy)^{-1})\\ &= (gx, gy, \theta_k(g)t\theta_l(g)^{-1}).\end{aligned}$$

Since X' is profinite, this implies that the unitary representation $\pi|_{\Gamma'}$ is contained in a direct sum of finite representations (of the form $\Gamma' \curvearrowright X_n \times X_n \times \Lambda$, $g(x, y, t) = (gx, gy, \theta_k(g)t\theta_l(g)^{-1})$). Since Γ' has finite index in Γ, the unitary representation $\pi \subset \mathrm{Ind}_{\Gamma'}^{\Gamma}(\pi|_{\Gamma'})$ is contained in a direct sum of finite representations. This completes the proof. □

The following two lemmas are well-known, but we include the proof for the reader's convenience.

Lemma 6.4. *Let $\Gamma \geq \Delta_1 \geq \Delta_2 \geq \cdots$ be a decreasing sequence of finite index normal subgroups. Then, the left-and-right action $\Gamma \times \Gamma \curvearrowright \varprojlim \Gamma/\Delta_n$ is essentially-free if and only if $\lim_n |Z_n(g)|/|\Gamma/\Delta_n| = 0$ for every $g \in \Gamma$ with $g \neq e$, where $Z_n(g)$ is the centralizer group of g in Γ/Δ_n.*

Proof. The 'only if' part is trivial. We prove the 'if' part. Note that the condition implies that $\bigcap \Delta_n = \{e\}$. Let $(g, h) \in \Gamma \times \Gamma$ and observe that

$$|\{x \in \varprojlim \Gamma/\Delta_n : (g, h)x = x\}| = \lim_n \frac{|\{x \in \Gamma/\Delta_n : gxh^{-1} = x\}|}{|\Gamma/\Delta_n|}.$$

If $g = e$, then (g,h) acts freely unless $h = e$, too. Thus, let $g \neq e$. If $x, y \in \Gamma/\Delta_n$ are such that $gxh^{-1} = x$ and $gyh^{-1} = y$, then one has $gxy^{-1}g^{-1} = xy^{-1}$, i.e., $xy^{-1} \in Z_n(g)$. It follows that $|\{x \in \Gamma/\Delta_n : gxh^{-1} = x\}| \leq |Z_n(g)|$. □

Lemma 6.5. *Let F be a finite field. Then, for every $g \in \mathrm{PSL}(2,F)$ with $g \neq e$, one has $|Z(g)|/|\mathrm{PSL}(2,F)| \leq 2/(|F|-1)$.*

Proof. Since the characteristic polynomial of g is quadratic, it can be factorized in some quadratic extension $\tilde{F}$ of F. Thus g is conjugate to a Jordan normal form in $\mathrm{PSL}(2,\tilde{F})$. Now, it is not hard to see that the centralizer of g in $\mathrm{PSL}(2,\tilde{F})$ has cardinality at most $|\tilde{F}| = |F|^2$. On the other hand, it is well-known that $|\mathrm{SL}(2,F)| = |F|(|F|^2-1)$. □

Proof of Corollary C. Since $\mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ is an irreducible lattice in $\mathrm{SL}(2,\mathbb{R})^2$, it has property (τ) (see Section 4.3 in [Lu]) and the property $(\mathrm{HH})^+$ (cf. Theorem 2.3). By the above lemmas, the action $\Gamma \curvearrowright X$ is essentially-free. Indeed, consider the homomorphism from $\mathrm{PSL}(2,(\mathbb{Z}/p_1\cdots p_n\mathbb{Z})[\sqrt{2}])$ onto $\mathrm{PSL}(2,F)$, where F is the field either $\mathbb{Z}/p_n\mathbb{Z}$ or $(\mathbb{Z}/p_n\mathbb{Z})[\sqrt{2}]$, depending on whether the equation $x^2 = 2$ is solvable in $\mathbb{Z}/p_n\mathbb{Z}$ or not; and apply Lemma 6.5 at $\mathrm{PSL}(2,F)$. Therefore, by Corollary A, $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X)\rtimes\Gamma$. It follows that the isomorphism of von Neumann algebras $L^\infty(X)\rtimes\Gamma \cong (L^\infty(Y)\rtimes\Lambda)^t$ gives rise to a stable orbit equivalence between $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$. The growth condition of Theorem C is satisfied because p_k's are mutually distinct primes and $\mathrm{PSL}(2,(\mathbb{Z}/p_1\cdots p_n\mathbb{Z})[\sqrt{2}]) \cong \prod \mathrm{PSL}(2,(\mathbb{Z}/p_k\mathbb{Z})[\sqrt{2}])$. Therefore, Theorem C is applicable to the orbit equivalence cocycle $\alpha\colon \Gamma \times X \to \Lambda$. For the rest of the proof, see [Io]. □

7. II_1-factors with more than one Cartan subalgebra

Proof of Theorem D. Since $H \curvearrowright^\sigma X$ is an ergodic and profinite action, one has $X = \varprojlim H/H_n$ for some decreasing sequence $H = H_0 \supset H_1 \supset \cdots$ of finite-index subgroups of H such that $\bigcap H_n = \{e\}$. Recall that a function f is called an eigenfunction of H if there is a character χ on H such that $\sigma_h(f) = \chi(h)f$ for every $h \in H$. We observe that every unitary eigenfunction normalizes $L(H)$ in $L^\infty(X) \rtimes H$, and that $L^\infty(X)$ is spanned by unitary eigenfunctions since $L^\infty(H/H_n)$ is spanned by characters. This proves that $L(H)$ is regular in $L^\infty(X)\rtimes\Gamma$. To prove that $L(H)$ is maximal abelian, let $a \in L(H)' \cap L^\infty(X)\rtimes\Gamma$ be given and $a = \sum_{g\in\Gamma} a_g u_g$ be the Fourier expansion. Then, $[a, u_h] = 0$ implies $\sigma_h(a_g) = a_{hgh^{-1}}$ for all $g \in \Gamma$ and $h \in H$. In particular, one has $\|a_{hgh^{-1}}\|_2 = \|a_g\|_2$. Since $\sum_g \|a_g\|_2^2 = \|a\|_2^2 < \infty$, the relative ICC condition implies that $a_g = 0$ for all $g \notin H$. But for $g \in H$, ergodicity of $H \curvearrowright X$ implies that $a_g \in \mathbb{C}1$. This proves $a \in L(H)$.

For the second assertion, recall that weak compactness is an invariant of a Cartan subalgebra (Proposition 3.4 in [OP]). We prove that $L(H)$ is not weakly

compact in $L^\infty(X) \rtimes \Gamma$. Suppose by contradiction that it is weakly compact. Then, by Proposition 3.2 in [OP], there is a state φ on $\mathbb{B}(\ell^2(H))$ which is invariant under the $H \rtimes \Gamma$-action. Restricting φ to $\ell^\infty(H)$, we obtain an $H \rtimes \Gamma$-invariant mean. This contradicts the assumption. $\square$

Corollary D is an immediate consequence of Theorem D. Here we give another example for which Theorem D applies. Let K be a residually-finite additive group such that $|K| > 1$, and Γ_0 be a residually-finite non-amenable group. The wreath product $\Gamma = K \wr \Gamma_0$ is defined to be the semidirect product of $H = \bigoplus_{\Gamma_0} K$ by the shift action of Γ_0. Then, there is a decreasing sequence $H_0 \supset H_1 \supset \cdots$ of Γ_0-invariant finite-index subgroups of H such that $\bigcap H_n = \{0\}$. Indeed, let $K_0 \supset K_1 \supset \cdots$ (resp. $\Gamma_{0,0} \supset \Gamma_{0,1} \supset \cdots$) be finite-index subgroups of K (resp. Γ_0) such that $\bigcap K_n = \{0\}$ (resp. $\bigcap \Gamma_{0,n} = \{e\}$). Then, the "augmentation subgroups"

$$H_n = \{(a_g)_{g\in\Gamma_0} \in H : \sum_{h\in\Gamma_{0,n}} a_{gh} \in K_n \text{ for all } g \in \Gamma_0\},$$

which is the kernel of the homomorphism onto $\bigoplus_{\Gamma_0/\Gamma_{0,n}} K/K_n$, satisfy the required conditions. It follows from Theorem D that the II_1-factor

$$L^\infty(\varprojlim H/H_n) \rtimes \Gamma$$

has two non-conjugate Cartan subalgebras, namely $L(H)$ and $L^\infty(\varprojlim H/H_n)$.

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